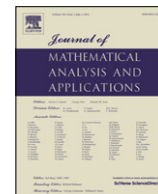


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The Pexider type generalization of the Minkowski inequality

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ABSTRACT

Let (Ω, Σ, μ) be a measure space such that $0 < \mu(A) < 1 < \mu(B) < \infty$ for some $A, B \in \Sigma$. The following converse Minkowski inequality theorem is proved in Matkowski (2008) [4]. If $\varphi, \psi, \gamma : (0, \infty) \rightarrow (0, \infty)$ are bijective, φ is increasing, and

$$\varphi^{-1} \left(\int_{\Omega(\mathbf{x}+\mathbf{y})} \varphi \circ (\mathbf{x} + \mathbf{y}) d\mu \right) \leq \psi^{-1} \left(\int_{\Omega(\mathbf{x})} \psi \circ \mathbf{x} d\mu \right) + \gamma^{-1} \left(\int_{\Omega(\mathbf{y})} \gamma \circ \mathbf{y} d\mu \right)$$

for all nonnegative μ -integrable simple functions $\mathbf{x}, \mathbf{y} : \Omega \rightarrow \mathbb{R}$ (where $\Omega(\mathbf{x})$ stands for the support of \mathbf{x}), then there exists a real $p \geq 1$ such that

$$\frac{\varphi(t)}{\varphi(1)} = \frac{\psi(t)}{\psi(1)} = \frac{\gamma(t)}{\gamma(1)} = t^p.$$

In the present paper we show that if, in the basic measure space, there is no $A \in \Sigma$ such that either $1 < \mu(A) < \infty$ or $0 < \mu(A) < 1$, then there are some broad classes of non-power functions which satisfy the above Minkowski type inequality. Moreover we prove that, in the converse of the Minkowski inequality theorem, the assumption of the increasing monotonicity of φ is essential.

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0. Introduction

For a measure space (Ω, Σ, μ) , $S = S(\Omega, \Sigma, \mu)$ denotes the linear real space of all μ -integrable simple functions $\mathbf{x} : \Omega \rightarrow \mathbb{R}$. Let $S_+ := \{\mathbf{x} \in S : \mathbf{x} \geq 0\}$. For the support of $\mathbf{x} \in S$ put

$$\Omega(\mathbf{x}) := \{\omega \in \Omega : \mathbf{x}(\omega) \neq 0\}.$$

Let $\varphi : (0, \infty) \rightarrow (0, \infty)$ be an arbitrary bijection (that is one-to-one and onto: $\varphi((0, \infty)) = (0, \infty)$). Since, for any $\mathbf{x} \in S_+$, the restriction $\varphi \circ \mathbf{x}|_{\Omega(\mathbf{x})} \in S_+$, the functional $\mathbb{P}_\varphi : S \rightarrow [0, \infty)$ given by the formula

$$\mathbb{P}_\varphi(\mathbf{x}) := \begin{cases} \varphi^{-1} \left(\int_{\Omega(\mathbf{x})} \varphi \circ |\mathbf{x}| d\mu \right) & \text{if } \mu(\Omega(\mathbf{x})) > 0 \\ 0 & \text{if } \mu(\Omega(\mathbf{x})) = 0, \end{cases}$$

is correctly defined.

By the Minkowski inequality:

if $\varphi(t) = \varphi(1)t^p$ for some $p \geq 1$, then

$$\mathbb{P}_\varphi(\mathbf{x} + \mathbf{y}) \leq \mathbb{P}_\varphi(\mathbf{x}) + \mathbb{P}_\varphi(\mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in S_+(\Omega, \Sigma, \mu). \quad (\text{M})$$

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In [1] (cf. also [2]) it has been shown that, in general, under some rather weak regularity assumptions of φ (or without any), the converse implication holds if,

$$\text{there are } A, B \in \Sigma \text{ such that } 0 < \mu(A) < 1 < \mu(B) < \infty. \quad (*)$$

(If the range of the measure is enough rich, no regularity assumption is needed [3].)

In [4] the following result has been proved. Suppose that there are $A, B \in \Sigma$ such that $(*)$ holds true and $\varphi, \psi, \gamma : (0, \infty) \rightarrow (0, \infty)$ are bijective. If φ is increasing, then

$$\mathbb{P}_\varphi(\mathbf{x} + \mathbf{y}) \leq \mathbb{P}_\psi(\mathbf{x}) + \mathbb{P}_\gamma(\mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in S_+(\Omega, \Sigma, \mu), \quad (\text{P})$$

if, and only if, there exists a real $p \geq 1$ such that

$$\frac{\varphi(t)}{\varphi(1)} = \frac{\psi(t)}{\psi(1)} = \frac{\gamma(t)}{\gamma(1)} = t^p, \quad t > 0.$$

Inequality (P) is called a “Pexiderization” of the Minkowski inequality (M) as, for the first time, an analogous procedure was applied in 1903 by Pexider [5] for the Cauchy functional equation. Let us note that, under the basic condition $(*)$, and the increasing monotonicity of φ , the above converse of the Minkowski inequality theorem remains true, if (P) is replaced by the inequality

$$\mathbb{P}_\varphi\left(\sum_{i=1}^n \mathbf{x}_i\right) \leq \sum_{i=1}^n \mathbb{P}_{\psi_i}(\mathbf{x}_i), \quad \mathbf{x}_i \in S_+(\Omega, \Sigma, \mu), \quad i = 1, \dots, n,$$

where $n \in \mathbb{N}$, $n \geq 2$, is arbitrarily fixed (cf. [4]).

In Section 2 we show that, in these results, the assumption of the increasing monotonicity of φ is indispensable.

Condition $(*)$ plays here a crucial role. Note that the measure space (Ω, Σ, μ) does not satisfy $(*)$, iff either

(I) for every $A \in \Sigma$, we have $\mu(A) = 0$ or $\mu(A) \geq 1$;

or

(II) for every $A \in \Sigma$, we have $\mu(A) \leq 1$ or $\mu(A) = \infty$.

In the present paper, in each of these two cases we indicate some broad classes of *non-power functions* satisfying inequality (P). In case (I), the results obtained in Section 3 extend an earlier Mulholland inequality [6] (cf. also [7,8]) as well as its generalizations [1,2,4] concerning inequality (M). It turns out that the convexity and geometrical convexity of φ plays a meaningful role.

Section 4 is devoted to the “Pexiderized” Minkowski inequality in case (II). In particular we show that if (Ω, Σ, μ) is a non-trivial probability measure space, inequality (P) is equivalent to concavity of the function

$$(0, \infty)^2 \ni (s, t) \mapsto \varphi(\psi^{-1}(s) + \gamma^{-1}(t)).$$

1. Preliminaries

Let (Ω, Σ, μ) be a measure space. Denote by χ_A the characteristic function of a set A . For $A, B \in \Sigma$, $A \cap B = \emptyset$, we put

$$S_+(A, B) := \{x_1 \chi_A + x_2 \chi_B \in S_+ : x_1, x_2 > 0\}$$

and

$$a := \mu(A), \quad b := \mu(B).$$

Let $\varphi, \psi, \gamma : (0, \infty) \rightarrow (0, \infty)$ be arbitrary bijections. If $\mathbf{x} := x_1 \chi_A + x_2 \chi_{B \setminus A} \in S_+(A, B)$ then, by the definition of \mathbb{P}_φ , we have

$$\mathbb{P}_\varphi(\mathbf{x}) = \varphi^{-1}(a\varphi(x_1 + y_1) + b\varphi(x_2 + y_2)).$$

It follows that the inequality

$$\mathbb{P}_\varphi(\mathbf{x} + \mathbf{y}) \leq \mathbb{P}_\psi(\mathbf{x}) + \mathbb{P}_\gamma(\mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in S_+(A, B), \quad (1.1)$$

where

$$\mathbf{x} := x_1 \chi_A + x_2 \chi_{B \setminus A}, \quad \mathbf{y} := y_1 \chi_A + y_2 \chi_{B \setminus A},$$

for some $x_1, x_2, y_1, y_2 > 0$, (that is, inequality (P) restricted to the set $S_+(A, B)$) can be written in the following more explicit form:

$$\begin{aligned} \varphi^{-1}(a\varphi(x_1 + y_1) + b\varphi(x_2 + y_2)) &\leq \psi^{-1}(a\psi(x_1) + b\psi(x_2)) + \gamma^{-1}(a\gamma(y_1) + b\gamma(y_2)), \\ x_1, x_2, y_1, y_2 &> 0. \end{aligned} \quad (1.2)$$

We shall need the following obvious.

Remark 1. Let A, B be arbitrary disjoint nonempty sets and $a, b > 0$ real numbers. Then (Ω, Σ, μ) with $\Omega := A \cup B$, $\Sigma = \{\emptyset, A, B, A \cup B\}$ and μ defined by $\mu(A) := a$, $\mu(B) := b$ is a measure space. Moreover $S_+(\Omega, \Sigma, \mu) = S_+(A, B)$ and inequality (1.2) (as well as (1.1)) is equivalent to the inequality

$$\mathbb{P}_\varphi(\mathbf{x} + \mathbf{y}) \leq \mathbb{P}_\psi(\mathbf{x}) + \mathbb{P}_\gamma(\mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in S_+(\Omega, \Sigma, \mu).$$

2. Increasing monotonicity of φ in the converse of the Minkowski inequality theorem is essential

Let us quote the following [4] (Theorem 2).

Theorem 1. Let (Ω, Σ, μ) be a measure space with two sets $A, B \in \Sigma$ such that condition $(*)$ is fulfilled. Suppose that $\varphi, \psi, \gamma : (0, \infty) \rightarrow (0, \infty)$ are bijective functions and φ is strictly increasing. Then inequality (P) holds if, and only if, there is $p \geq 1$ such that

$$\frac{\varphi(t)}{\varphi(1)} = \frac{\psi(t)}{\psi(1)} = \frac{\gamma(t)}{\gamma(1)} = t^p, \quad t > 0.$$

To show that the assumption of the increasing monotonicity of φ in this converse of the Minkowski inequality is essential we prove the following.

Proposition 1. Let the real numbers $a, b > 0$ be arbitrarily fixed. If the bijections $\varphi, \psi, \gamma : (0, \infty) \rightarrow (0, \infty)$ such that

- (i) φ is decreasing and convex;
- (ii) ψ and γ are increasing and convex;
- (iii) the functions

$$f := \varphi^{-1} \circ ((a+b)\varphi), \quad g := \psi^{-1} \circ ((a+b)\psi), \quad h := \gamma^{-1} \circ ((a+b)\gamma)$$

satisfy the Pexider type subadditivity condition

$$f(s+t) \leq g(s) + h(t), \quad s, t > 0,$$

then inequality (1.2) holds true.

Proof. Replacing s by $\psi^{-1}(s)$, t by $\gamma^{-1}(t)$, and making use of decreasing monotonicity of φ^{-1} , we can write condition (iii) in the following equivalent form

$$\varphi(\psi^{-1}((a+b)s) + \gamma^{-1}((a+b)t)) \leq (a+b)\varphi(\psi^{-1}(s) + \gamma^{-1}(t)), \quad s, t > 0.$$

By (ii) the functions ψ^{-1} and γ^{-1} are concave. It follows that the functions

$$(0, \infty) \ni u \mapsto \psi^{-1}((a+b)u), \quad (0, \infty) \ni u \mapsto \gamma^{-1}((a+b)u)$$

are also concave. Therefore

$$\begin{aligned} \psi^{-1}(ax_1 + by_1) + \gamma^{-1}(ax_1 + by_1) &= \psi^{-1}\left((a+b)\left(\frac{a}{a+b}x_1 + \frac{b}{a+b}y_1\right)\right) \\ &\quad + \gamma^{-1}\left((a+b)\left(\frac{a}{a+b}x_1 + \frac{b}{a+b}y_1\right)\right) \\ &\geq \frac{a}{a+b}[\psi^{-1}((a+b)x_1) + \gamma^{-1}((a+b)x_2)] \\ &\quad + \frac{b}{a+b}[\psi^{-1}((a+b)y_1) + \gamma^{-1}((a+b)y_2)] \end{aligned}$$

for all $x_1, x_2, y_1, y_2 > 0$. By (i) the function φ is decreasing and convex. Hence, using in turn: the monotonicity of φ , the convexity of φ and the above equivalent form of condition (iii), we get

$$\begin{aligned} \varphi[\psi^{-1}(ax_1 + by_1) + \gamma^{-1}(ax_1 + by_1)] &\leq \varphi\left(\frac{a}{a+b}[\psi^{-1}((a+b)x_1) + \gamma^{-1}((a+b)x_2)]\right. \\ &\quad \left.+ \frac{b}{a+b}[\psi^{-1}((a+b)y_1) + \gamma^{-1}((a+b)y_2)]\right) \\ &\leq \frac{a}{a+b}\varphi[\psi^{-1}((a+b)x_1) + \gamma^{-1}((a+b)x_2)] \\ &\quad + \frac{b}{a+b}\varphi[\psi^{-1}((a+b)y_1) + \gamma^{-1}((a+b)y_2)] \\ &\leq a\varphi[\psi^{-1}(x_1) + \gamma^{-1}(x_2)] + b\varphi[\psi^{-1}(y_1) + \gamma^{-1}(y_2)] \end{aligned}$$

for all $x_1, x_2, y_1, y_2 > 0$.

Replacing here x_1, x_2, y_1, y_2 respectively by $\psi(x_1), \psi(x_2), \gamma(y_1), \gamma(y_2)$ and making again use of the decreasing monotonicity of φ , we obtain the required inequality (1.2). This completes the proof. \square

Example. The functions $\varphi, \psi, \gamma : (0, \infty) \rightarrow (0, \infty)$ given by

$$\varphi(t) := \frac{1}{t}, \quad \psi(t) := t^2, \quad \gamma := e^t - 1, \quad t > 0,$$

are homeomorphisms of $(0, \infty)$ and, obviously, satisfy conditions (i) and (ii) of [Proposition 1](#). Moreover, for arbitrary $a, b > 0$ we have

$$f(t) = \frac{t}{a+b}, \quad g(t) = \sqrt{a+bt}, \quad h(t) = \log((a+b)(e^t - 1) + 1), \quad t > 0.$$

If $a + b \geq 1$, then we have $h(t) \geq t$ for all $t > 0$, and, consequently, condition (iii) is satisfied.

Proposition 2. Let $a > 0$ be arbitrarily fixed. If the bijections $\varphi, \psi, \gamma : (0, \infty) \rightarrow (0, \infty)$ such that

- (i) φ is decreasing;
- (ii) ψ and γ are increasing;
- (iii) the functions

$$f := \varphi^{-1} \circ (a\varphi), \quad g := \psi^{-1} \circ (a\psi), \quad h := \gamma^{-1} \circ (a\gamma)$$

satisfy the following Pexider type subadditivity condition

$$f(s+t) \leq g(s) + h(t), \quad s, t > 0,$$

then (sharp) inequality (1.2) holds for all $b > 0$ and $x_1, x_2, y_1, y_2 > 0$.

Proof. Take arbitrary $b, x_1, x_2, y_1, y_2 > 0$. Applying in turn: the decreasing monotonicity of φ^{-1} , condition (iii), the increasing monotonicity of ψ^{-1} and γ^{-1} , we obtain

$$\begin{aligned} \varphi^{-1}(a\varphi(x_1 + y_1) + b\varphi(x_2 + y_2)) &< \varphi^{-1}(a\varphi(x_1 + y_1)) \\ &< \psi^{-1}(a\psi(x_1)) + \gamma^{-1}(a\gamma(y_1)) \\ &< \psi^{-1}(a\psi(x_1) + b\psi(x_2)) + \gamma^{-1}(a\gamma(y_1) + b\gamma(y_2)), \end{aligned}$$

which completes the proof. \square

Remark 2. Let real numbers a, b be such that $0 < \min\{a, b\} < 1 < a + b$. Take a measure space (Ω, Σ, μ) with $\Sigma = \{\emptyset, \Omega, A, B\}$ where $B = \Omega \setminus A$, such that $\mu(A) = a$ and $\mu(B) = b$. Obviously, we have $S_+(\Omega, \Sigma, \mu) = S_+(A, B)$ (cf. [Remark 1](#)). The bijections φ, ψ, γ given in [Proposition 1](#) (or [Proposition 2](#)) with φ decreasing, need not be power functions, but they satisfy inequality (1.1). This proves that the increasing monotonicity of the function φ in [Theorem 1](#) is essential.

The significance of the increasing monotonicity of the functions φ, ψ, γ in [Theorem 1](#) explains also the following.

Proposition 3. Let $a, b > 0$ be fixed. Then there does not exist a triple of decreasing bijections $\varphi, \psi, \gamma : (0, \infty) \rightarrow (0, \infty)$ such that inequality (1.2) holds true for all $x_1, x_2, y_1, y_2 > 0$.

Proof. Suppose, for an indirect argument, that there exist decreasing bijections φ, ψ, γ of $(0, \infty)$ (so homeomorphisms) such that inequality (1.2) holds true. Of course, for all $y_1 > 0$, we have

$$\lim_{y_2 \rightarrow 0} \gamma^{-1}(a\gamma(y_1) + b\gamma(y_2)) = 0,$$

and, for all $x_2 > 0$,

$$\lim_{x_1 \rightarrow 0} \psi^{-1}(a\psi(x_1) + b\psi(x_2)) = 0.$$

Hence, letting y_2 tend to 0 in inequality (1.2), and making use of the continuity of the functions φ and φ^{-1} , we get

$$\varphi^{-1}(a\varphi(x_1 + y_1) + b\varphi(x_2)) \leq \psi^{-1}(a\psi(x_1) + b\psi(x_2))$$

for all $x_1, x_2, y_1 > 0$, and letting here $x_1 \rightarrow 0$, we get

$$\varphi^{-1}(a\varphi(y_1) + b\varphi(x_2)) \leq 0, \quad y_1, x_2 > 0,$$

that is a contradiction. \square

3. The Pexider type generalization of the Minkowski inequality in case (I)

We begin with some auxiliary notions and results.

A function $\varphi : (0, \infty) \rightarrow (0, \infty)$ is said to be (cf. [9]) *geometrically convex* if

$$\varphi(s^r t^{1-r}) \leq [\varphi(s)]^r [\varphi(t)]^{1-r}, \quad s, t > 0, \quad r \in (0, 1);$$

geometrically concave if the reversed inequality is satisfied, and geometrically affine if

$$\varphi(s^r t^{1-r}) = [\varphi(s)]^r [\varphi(t)]^{1-r}, \quad s, t > 0, r \in (0, 1).$$

Note the following properties of geometrically convex functions.

Properties.

- (i) $\varphi : (0, \infty) \rightarrow (0, \infty)$ is geometrically convex (respectively geometrically concave, geometrically affine) iff $\log \circ \varphi \circ \exp$ is convex (respectively concave, affine) in \mathbb{R} .
- (ii) If $\varphi : (0, \infty) \rightarrow (0, \infty)$ is geometrically convex, then φ'_- and φ'_+ , the left and right derivatives of φ exist, and the functions

$$(0, \infty) \ni t \mapsto \frac{\varphi'_-(t)}{\varphi(t)} t, \quad (0, \infty) \ni t \mapsto \frac{\varphi'_+(t)}{\varphi(t)} t$$

are increasing in $(0, \infty)$. Conversely, if φ is continuous, one-sided differentiable and one of the above functions is increasing, then φ is geometrically convex.

- (iii) Suppose that $\varphi : (0, \infty) \rightarrow (0, \infty)$ is bijective and increasing. Then φ is geometrically convex iff φ^{-1} is geometrically concave.
- (iv) Suppose that $\varphi : (0, \infty) \rightarrow (0, \infty)$ is bijective and decreasing. Then φ is geometrically convex iff φ^{-1} is geometrically convex.
- (v) If $\varphi : (0, \infty) \rightarrow (0, \infty)$ is geometrically convex and the function $(0, \infty) \ni t \mapsto \frac{\varphi(t)}{t}$ is increasing, then φ is convex.

The proofs are easy. For instance, to show (v) observe that

$$\varphi'_+(t) = \left(\frac{\varphi'_+(t)}{\varphi(t)} t \right) \left(\frac{\varphi(t)}{t} \right),$$

and, in view of (ii), the function φ'_+ is increasing in $(0, \infty)$, as the product of positive increasing functions.

Each of the lemmas below generalizes the Mulholland result [6].

Lemma 1. Suppose that $\varphi, \psi, \gamma : (0, \infty) \rightarrow (0, \infty)$ are bijective and such that

- (i) φ is geometrically convex;
- (ii) ψ'_- and γ'_- (or ψ'_+ and γ'_+) exist ψ'_- and γ'_- and the functions

$$(0, \infty) \ni u \mapsto \frac{\psi(u)}{u}, \quad (0, \infty) \ni u \mapsto \frac{\gamma(u)}{u}$$

are nondecreasing;

- (iii) the functions $\frac{\varphi}{\psi}$ and $\frac{\varphi}{\gamma}$ are nondecreasing.

Then, for arbitrarily fixed $s, t > 0$, the function $F : (0, s] \times (0, t] \rightarrow (0, \infty)$ given by

$$F(u, v) := \frac{tu\psi^{-1}(s) + sv\gamma^{-1}(t)}{st\varphi(\psi^{-1}(u) + \gamma^{-1}(v))}, \quad 0 < u \leq s, 0 < v \leq t, \quad (3.1)$$

admits the minimum at the point (s, t) , i.e.

$$0 < u \leq s, \quad 0 < v \leq t \implies F(u, v) \geq F(s, t).$$

Proof. Suppose, for instance, that ψ and γ are left-side differentiable. For the simplicity of notations, by φ', ψ', γ' we denote the left-side derivatives $\varphi'_-, \psi'_-, \gamma'_-$, respectively.

By (ii) the functions ψ and γ are increasing. This assumption implies also that $\psi' > 0$ and $\gamma' > 0$. In view of (ii), the left derivative $\varphi' = \varphi'_-$ exists in $(0, \infty)$. From (iii) we have

$$\frac{\psi'(u)}{\psi(u)} \leq \frac{\varphi'(u)}{\varphi(u)}, \quad \frac{\gamma'(u)}{\gamma(u)} \leq \frac{\varphi'(u)}{\varphi(u)}, \quad u > 0. \quad (3.2)$$

Each of these inequalities together with inequalities $\psi' > 0, \gamma' > 0$ implies that $\varphi' > 0$. It follows that φ, ψ, γ are increasing homeomorphisms of $(0, \infty)$. By the chain rule (for the left derivative), at every point of the set $(0, s] \times (0, t]$, there exists the left partial derivative of F and

$$\begin{aligned} \frac{\partial(\log \circ F)}{\partial u}(u, v) &= \frac{t\psi^{-1}(s)}{tu\psi^{-1}(u) + sv\gamma^{-1}(v)} - \frac{\varphi'(\psi^{-1}(u) + \gamma^{-1}(v))}{\varphi(\psi^{-1}(u) + \gamma^{-1}(v))} \frac{1}{\psi'(\psi^{-1}(u))}, \\ \frac{\partial(\log \circ F)}{\partial v}(u, v) &= \frac{s\gamma^{-1}(t)}{tu\psi^{-1}(u) + sv\gamma^{-1}(v)} - \frac{\varphi'(\psi^{-1}(u) + \gamma^{-1}(v))}{\varphi(\psi^{-1}(u) + \gamma^{-1}(v))} \frac{1}{\gamma'(\gamma^{-1}(v))}. \end{aligned}$$

Making use of the geometrical convexity of φ , property (ii) and (3.2), we get

$$\begin{aligned} \frac{\varphi'(\psi^{-1}(u) + \gamma^{-1}(v))}{\varphi(\psi^{-1}(u) + \gamma^{-1}(v))} \frac{1}{\psi'(\psi^{-1}(u))} &= \frac{\varphi'(\psi^{-1}(u) + \gamma^{-1}(v))}{\varphi(\psi^{-1}(u) + \gamma^{-1}(v))} (\psi^{-1}(u) + \gamma^{-1}(v)) \\ &\quad \times \frac{1}{\psi^{-1}(u) + \gamma^{-1}(v)} \frac{1}{\psi'(\psi^{-1}(u))} \\ &\geq \frac{\varphi'(\psi^{-1}(u))}{\varphi(\psi^{-1}(u))} \psi^{-1}(u) \frac{1}{\psi^{-1}(u) + \gamma^{-1}(v)} \frac{1}{\psi'(\psi^{-1}(u))} \\ &\geq \frac{\psi'(\psi^{-1}(u))}{\psi(\psi^{-1}(u))} \psi^{-1}(u) \frac{1}{\psi^{-1}(u) + \gamma^{-1}(v)} \frac{1}{\psi'(\psi^{-1}(u))} \\ &= \frac{\psi^{-1}(u)}{u(\psi^{-1}(u) + \gamma^{-1}(v))}, \end{aligned}$$

whence

$$\frac{\partial(\log \circ F)}{\partial u}(u, v) \leq \frac{t\psi^{-1}(s)}{tu\psi^{-1}(u) + sv\gamma^{-1}(v)} - \frac{\psi^{-1}(u)}{u(\psi^{-1}(u) + \gamma^{-1}(v))},$$

for all $0 < u \leq s, 0 < v \leq t$.

In the same way we can show that

$$\frac{\partial(\log \circ F)}{\partial v}(u, v) \leq \frac{s\gamma^{-1}(t)}{tu\psi^{-1}(u) + sv\gamma^{-1}(v)} - \frac{\gamma^{-1}(v)}{v(\psi^{-1}(u) + \gamma^{-1}(v))},$$

for all $0 < u \leq s, 0 < v \leq t$.

Now consider the following three types of curves

$$(I) \begin{cases} u(\tau) = k\tau \\ v(\tau) = m\tau \end{cases} \quad (k, m \geq 0); \quad (II) \begin{cases} u(\tau) = s \\ v(\tau) = \tau; \end{cases} \quad (III) \begin{cases} u(\tau) = \tau \\ v(\tau) = t \end{cases}$$

passing inside the set $(0, s] \times (0, t]$.

Let us note that to prove our lemma it is sufficient to show that in each of these three cases the function

$$h(\tau) := (\log \circ F)(u(\tau), v(\tau))$$

is nonincreasing.

Put $h' := h'_-$. For an arbitrary regular curve $(0, T] \ni \tau \mapsto (u(\tau), v(\tau))$ passing inside the set $(0, s] \times (0, t]$, applying the above inequalities, performing simple calculations, we obtain, for all $\tau \in (0, T]$

$$\begin{aligned} h'(\tau) &= \frac{\partial(\log \circ F)}{\partial u}(u(\tau), v(\tau)) u'(\tau) + \frac{\partial(\log \circ F)}{\partial v}(u(\tau), v(\tau)) v'(\tau) \\ &\leq \left[\frac{t\psi^{-1}(s)}{tu(\tau)\psi^{-1}(u(\tau)) + sv(\tau)\gamma^{-1}(v(\tau))} - \frac{\psi^{-1}(u(\tau))}{u(\tau)(\psi^{-1}(u(\tau)) + \gamma^{-1}(v(\tau)))} \right] u'(\tau) \\ &\quad + \left[\frac{s\gamma^{-1}(t)}{tu(\tau)\psi^{-1}(u(\tau)) + sv(\tau)\gamma^{-1}(v(\tau))} - \frac{\gamma^{-1}(v(\tau))}{v(\tau)(\psi^{-1}(u(\tau)) + \gamma^{-1}(v(\tau)))} \right] v'(\tau). \end{aligned}$$

Substituting here an arbitrary curve of type (I) we get $h'(\tau) \leq 0$ for all $\tau \in (0, T]$.

Substituting here an arbitrary curve of type (II), and taking into account that $u'(\tau) = 0, v'(\tau) = 1$, we hence get, for all $\tau \in (0, t]$,

$$\begin{aligned} h'(\tau) &\leq \frac{s\gamma^{-1}(t)}{tu(\tau)\psi^{-1}(u(\tau)) + sv(\tau)\gamma^{-1}(v(\tau))} - \frac{\gamma^{-1}(v(\tau))}{v(\tau)(\psi^{-1}(u(\tau)) + \gamma^{-1}(v(\tau)))} \\ &= \frac{s\gamma^{-1}(t)}{ts\psi^{-1}(s) + s\tau\gamma^{-1}(\tau)} - \frac{\gamma^{-1}(\tau)}{\tau(\psi^{-1}(s) + \gamma^{-1}(\tau))} \\ &= \frac{\psi^{-1}(s)[\gamma^{-1}(t)\tau - \gamma^{-1}(\tau)t] + \tau\gamma^{-1}(\tau)[\gamma^{-1}(t) - \gamma^{-1}(\tau)]}{[ts\psi^{-1}(s) + s\tau\gamma^{-1}(\tau)][\tau(\psi^{-1}(s) + \gamma^{-1}(\tau))]} \\ &= \frac{\psi^{-1}(s)t\tau\left[\frac{\gamma^{-1}(t)}{t} - \frac{\gamma^{-1}(\tau)}{\tau}\right] + \tau\gamma^{-1}(\tau)[\gamma^{-1}(t) - \gamma^{-1}(\tau)]}{[ts\psi^{-1}(s) + s\tau\gamma^{-1}(\tau)][\tau(\psi^{-1}(s) + \gamma^{-1}(\tau))]} \end{aligned}$$

Since γ^{-1} is increasing and, by assumption, the function $\tau \mapsto \frac{\gamma^{-1}(\tau)}{\tau}$ is increasing, the function being the restriction of the right-hand side of above inequality to the interval $(0, t]$ attains its maximum equal 0 at the point $\tau = t$. This proves that $h'(\tau) \leq 0$ for all $\tau \in (0, t]$.

We omit a similar reasoning in the case when the curve is of type (III). This completes the proof. \square

Note that the same argument permits to prove the following n -dimensional counterpart of Lemma 1:

Lemma 2. Let $n \in \mathbb{N}$, $n \geq 2$ be fixed. Suppose that $\varphi, \psi_1, \dots, \psi_n : (0, \infty) \rightarrow (0, \infty)$ are bijective and such that

- (i) φ is geometrically convex;
- (ii) for each $i \in \{1, \dots, n\}$, the function ψ_i is left-differentiable (or for each $i \in \{1, \dots, n\}$, the function ψ_i is right-differentiable) and the function

$$(0, \infty) \ni t \mapsto \frac{\psi_i(t)}{t}$$

is nondecreasing;

- (iii) for each $i \in \{1, \dots, n\}$, the function $\frac{\varphi}{\psi_i}$ is nondecreasing.

Then, for arbitrary real numbers $s_1, \dots, s_n > 0$, the function $F : (0, s_1] \times \dots \times (0, s_n] \rightarrow (0, \infty)$ given by

$$F(x_1, \dots, x_n) := \frac{s_1^{-1}x_1\psi_1^{-1}(s_1) + \dots + s_n^{-1}x_n\psi_n^{-1}(s_n)}{\varphi(\psi_1^{-1}(x_1) + \dots + \psi_n^{-1}(x_n))}$$

admits the minimum at the point (s_1, \dots, s_n) , i.e.

$$0 < x_1 \leq s_1, \dots, 0 < x_n \leq s_n \implies F(x_1, \dots, x_n) \geq F(s_1, \dots, s_n).$$

Similarly as Lemma 1 we can prove the following

Lemma 3. Suppose that $\varphi, \psi, \gamma : (0, \infty) \rightarrow (0, \infty)$ are bijective and such that

- (i) φ is geometrically concave;
- (ii) ψ'_- and γ'_- (or ψ'_+ and γ'_+) exist φ'_- and γ'_- and the functions

$$(0, \infty) \ni u \longrightarrow \frac{\psi(u)}{u}, \quad (0, \infty) \ni u \longrightarrow \frac{\gamma(u)}{u}$$

are nonincreasing;

- (iii) the functions $\frac{\varphi}{\psi}$ and $\frac{\varphi}{\gamma}$ are nonincreasing.

Then, for arbitrarily fixed $s, t > 0$, the function $F : (0, s] \times (0, t] \rightarrow (0, \infty)$ given by

$$F(u, v) := \frac{tu\psi^{-1}(s) + sv\gamma^{-1}(t)}{st\varphi(\psi^{-1}(u) + \gamma^{-1}(v))}, \quad 0 < u \leq s, \quad 0 < v \leq t,$$

admits the maximum at the point (s, t) .

Let us also note the n th dimensional counterpart of Lemma 3.

Lemma 4. Let $n \in \mathbb{N}$, $n \geq 2$ be fixed. Suppose that $\varphi, \psi_1, \dots, \psi_n : (0, \infty) \rightarrow (0, \infty)$ are bijective and such that

- (i) φ is geometrically concave;
- (ii) for each $i \in \{1, \dots, n\}$, the function ψ_i is left-differentiable (or for each $i \in \{1, \dots, n\}$, the function ψ_i is right-differentiable) and the function

$$(0, \infty) \ni t \mapsto \frac{\psi_i(t)}{t}$$

is nonincreasing;

- (iii) for each $i \in \{1, \dots, n\}$, the function $\frac{\varphi}{\psi_i}$ is nonincreasing.

Then, for arbitrary real numbers $s_1, \dots, s_n > 0$, the function $F : (0, s_1] \times \dots \times (0, s_n] \rightarrow (0, \infty)$ given by

$$F(x_1, \dots, x_n) := \frac{s_1^{-1}x_1\psi_1^{-1}(s_1) + \dots + s_n^{-1}x_n\psi_n^{-1}(s_n)}{\varphi(\psi_1^{-1}(x_1) + \dots + \psi_n^{-1}(x_n))}$$

admits the maximum at the point (s_1, \dots, s_n) .

Now we can prove the existence of the broad classes of triples (φ, ψ, γ) of non-power functions satisfying inequality (P) in the case when the underlying measure space (Ω, Σ, μ) satisfies the condition

- (I) for every $A \in \Sigma$, we have $\mu(A) = 0$ or $\mu(A) \geq 1$.

Theorem 2. Let (Ω, Σ, μ) be a measure space with $\Sigma = \{\emptyset, \Omega, A, \Omega \setminus A\}$ for some $A \subset \Omega$ such that $\mu(A), \mu(\Omega \setminus A)$ are positive real numbers, and

$$\min\{\mu(A), \mu(\Omega \setminus A)\} \geq 1.$$

Suppose that $\varphi, \psi, \gamma : (0, \infty) \rightarrow (0, \infty)$ are bijective and such that

- (i) φ is geometrically convex;
- (ii) φ'_- and γ'_- (or φ'_+ and γ'_+) exist φ'_- and γ'_- and the functions

$$(0, \infty) \ni u \longrightarrow \frac{\varphi(u)}{u}, \quad (0, \infty) \ni u \longrightarrow \frac{\gamma(u)}{u}$$

are nondecreasing;

- (iii) the functions $\frac{\varphi}{\psi}$ and $\frac{\varphi}{\gamma}$ are nondecreasing;
- (iv) for each $r \in \mu(\Sigma) \setminus \{0\}$, the functions

$$f_r := \varphi^{-1} \circ (r\varphi), \quad g_r := \psi^{-1} \circ (r\psi), \quad h_r := \gamma^{-1} \circ (r\gamma)$$

satisfy the “Pexiderized” subadditivity condition:

$$f_r(u + v) \leq g_r(u) + h_r(v), \quad u, v > 0.$$

Then

$$\mathbb{P}_\varphi(\mathbf{x} + \mathbf{y}) \leq \mathbb{P}_\psi(\mathbf{x}) + \mathbb{P}_\gamma(\mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in S_+(\Omega, \Sigma, \mu).$$

Proof. Take arbitrary $\mathbf{x}, \mathbf{y} \in S_+(\Omega, \Sigma, \mu)$. Then such that

$$\mathbf{x} = x_1 \chi_A + x_2 \chi_B, \quad \mathbf{y} = y_1 \chi_A + y_2 \chi_B,$$

for some $x_1, x_2, y_1, y_2 > 0$, where $B := \Omega \setminus A$. Put

$$s := x_1 + x_2, \quad t := y_1 + y_2,$$

and define $F : (0, s] \times (0, t] \rightarrow (0, \infty)$ by (3.1). According to Lemma 1 we have $F(x_1, x_2) \geq F(s, t)$, i.e.

$$\frac{tx_1\psi^{-1}(s) + sy_1\gamma^{-1}(t)}{st(\psi^{-1}(x_1) + \gamma^{-1}(y_1))} \geq \frac{\psi^{-1}(s) + \gamma^{-1}(t)}{\varphi(\psi^{-1}(s) + \gamma^{-1}(t))}$$

and

$$\frac{tx_2\psi^{-1}(s) + sy_2\gamma^{-1}(t)}{st(\psi^{-1}(x_2) + \gamma^{-1}(y_2))} \geq \frac{\psi^{-1}(s) + \gamma^{-1}(t)}{\varphi(\psi^{-1}(s) + \gamma^{-1}(t))}.$$

Multiplying the first of the above inequalities by $\varphi(\psi^{-1}(x_1) + \gamma^{-1}(y_1))$, the second one by $\varphi(\psi^{-1}(x_2) + \gamma^{-1}(y_2))$, adding the respective sides of the resulting inequalities, we get

$$\begin{aligned} & \frac{t(x_1 + x_2)\psi^{-1}(s) + s(y_1 + y_2)\gamma^{-1}(t)}{st} \\ & \geq \frac{\psi^{-1}(s) + \gamma^{-1}(t)}{\varphi(\psi^{-1}(s) + \gamma^{-1}(t))} [\varphi(\psi^{-1}(x_1) + \gamma^{-1}(y_1)) + \varphi(\psi^{-1}(x_2) + \gamma^{-1}(y_2))] \end{aligned}$$

whence, making use of the definition of the numbers s and t , we obtain

$$\varphi(\psi^{-1}(x_1 + y_1) + \gamma^{-1}(x_2 + y_2)) \geq \varphi(\psi^{-1}(x_1) + \gamma^{-1}(x_2)) + \varphi(\psi^{-1}(y_1) + \gamma^{-1}(y_2))$$

for arbitrary $x_1, x_2, y_1, y_2 > 0$.

Put

$$a := \mu(A), \quad b := \mu(\Omega \setminus A).$$

Let $r \in \{a, b\}$. From the “Pexiderized” subadditivity assumed in (iv) we have

$$\begin{aligned} \varphi^{-1} \circ (r\varphi)(u + v) & \leq \psi^{-1} \circ (r\psi)(u) + \gamma^{-1} \circ (r\gamma)(v), \quad u, v > 0, \\ \varphi^{-1}(r(\varphi(u + v))) & \leq \psi^{-1}(r\psi(u)) + \gamma^{-1}(r\gamma(v)), \quad r \in \{a, b\}, u, v > 0. \end{aligned}$$

Replacing u by $\psi^{-1}(u)$ and v by $\gamma^{-1}(v)$, we get

$$\varphi^{-1}(r(\varphi(\psi^{-1}(u) + \gamma^{-1}(v)))) \leq \psi^{-1}(ru) + \gamma^{-1}(rv), \quad r \in \{a, b\}, u, v > 0,$$

whence, taking into account that φ is increasing (that is a consequence of (ii) and (iii)), we obtain

$$\varphi(\psi^{-1}(ru) + \gamma^{-1}(rv)) \geq r\varphi(\psi^{-1}(u) + \gamma^{-1}(v)), \quad r \in \{a, b\}, \quad u, v > 0.$$

Making use of the previous inequality, we hence get, for all $x_1, x_2, y_1, y_2 > 0$,

$$\begin{aligned} \varphi(\psi^{-1}(ax_1 + by_1) + \gamma^{-1}(ax_2 + by_2)) &\geq \varphi(\psi^{-1}(ax_1) + \gamma^{-1}(ax_2)) + \varphi(\psi^{-1}(by_1) + \gamma^{-1}(by_2)) \\ &\geq a\varphi(\psi^{-1}(x_1) + \gamma^{-1}(x_2)) + b\varphi(\psi^{-1}(y_1) + \gamma^{-1}(y_2)). \end{aligned}$$

Since $x_1, x_2, y_1, y_2 > 0$ can be taken arbitrarily, replacing here x_i by $\psi(x_i)$, y_i by $\gamma(y_i)$ ($i = 1, 2$), and making use of the increasing monotonicity of the function φ^{-1} , we obtain

$$\begin{aligned} \mathbb{P}_\psi(\mathbf{x}) + \mathbb{P}_\gamma(\mathbf{y}) &= \psi^{-1}(a\psi(x_1) + b\psi(x_2)) + \gamma^{-1}(a\gamma(y_1) + b\gamma(y_2)) \\ &\geq \varphi^{-1}(a\varphi(x_1 + y_1) + b\varphi(x_2 + y_2)) = \mathbb{P}_\varphi(\mathbf{x} + \mathbf{y}), \end{aligned}$$

which completes the proof. \square

Remark 3. The assumption (iv) of Theorem 2 is equivalent to the condition that for each $r \in \{\mu(A), \mu(\Omega \setminus A)\}$,

$$r\varphi(\psi^{-1}(u) + \gamma^{-1}(v)) \leq \varphi(\psi^{-1}(ru) + \gamma^{-1}(rv)), \quad u, v > 0.$$

This assumption is satisfied if $\mu(A) = \mu(\Omega \setminus A) = 1$.

Arguing similarly as in the proof of Theorem 2 and applying Lemma 2, we can prove the following

Theorem 3. Let $n \in \mathbb{N}$, $n \geq 2$ be fixed and (Ω, Σ, μ) be a measure space such that $\Sigma = \{\emptyset, A_1, \dots, A_n\}$ for some pairwise disjoint sets $A_i \subset \Omega$, $i = 1, \dots, n$, of positive finite measures and such that

$$\min \{\mu(A_i) : i = 1, \dots, n\} \geq 1.$$

Suppose that $\varphi, \psi_1, \dots, \psi_n : (0, \infty) \rightarrow (0, \infty)$ are bijective and such that

- (i) φ is geometrically convex;
- (ii) for each $i \in \{1, \dots, n\}$, the function ψ_i is left-differentiable (or for each $i \in \{1, \dots, n\}$, the function ψ_i is right-differentiable) and the function

$$(0, \infty) \ni t \longmapsto \frac{\psi_i(t)}{t}$$

is nondecreasing;

- (iii) for each $i \in \{1, \dots, n\}$, the function $\frac{\varphi}{\psi_i}$ is nondecreasing;
- (iv) for each $r \in \mu(\Sigma) \setminus \{0\}$, the functions

$$f_r := \varphi^{-1} \circ (r\varphi); \quad g_{i,r} := \psi_i^{-1} \circ (r\psi_i), \quad i = 1, \dots, n,$$

satisfy the “Pexiderized” subadditivity condition:

$$f_r(u_1 + \dots + u_n) \leq g_{1,r}(u_1) + \dots + g_{n,r}(u_n), \quad u_1, \dots, u_n > 0.$$

Then

$$\mathbb{P}_\varphi(\mathbf{x}_1 + \dots + \mathbf{x}_n) \leq \mathbb{P}_\varphi(\mathbf{x}_1) + \dots + \mathbb{P}_\varphi(\mathbf{x}_n), \quad \mathbf{x}_1, \dots, \mathbf{x}_n \in S_+(\Omega, \Sigma, \mu).$$

Remark 4. The assumption (iv) of the above theorem is equivalent to the inequality

$$r\varphi(\psi^{-1}(u) + \gamma^{-1}(v)) \leq \varphi(\psi^{-1}(ru) + \gamma^{-1}(rv)), \quad r \in \mu(\Sigma) \setminus \{0\}, \quad u, v > 0,$$

and, obviously, it is satisfied in the case when $\mu(A_1) = \dots = \mu(A_n) = 1$.

At the end of this section we prove the following

Proposition 4. Let (Ω, Σ, μ) be a measure space with $\Sigma = \{\emptyset, \Omega, A, \Omega \setminus A\}$ for some $A \subset \Omega$ such that $\mu(A), \mu(\Omega \setminus A)$ are positive real numbers, and

$$\min \{\mu(A), \mu(\Omega \setminus A)\} \geq 1.$$

If $\varphi, \psi, \gamma : (0, \infty) \rightarrow (0, \infty)$ are homeomorphisms such that

- (i) φ is decreasing;
 - (ii) $\psi = \gamma$ is increasing;
- then

$$\mathbb{P}_\varphi(\mathbf{x} + \mathbf{y}) < \mathbb{P}_\psi(\mathbf{x}) + \mathbb{P}_\gamma(\mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in S_+(\Omega, \Sigma, \mu),$$

(that is sharp inequality is satisfied).

Proof. Take arbitrary $\mathbf{x}, \mathbf{y} \in S_+(\Omega, \Sigma, \mu)$. Then there exist $x_1, x_2, y_1, y_2 > 0$ such that

$$\mathbf{x} = x_1 \chi_A + x_2 \chi_B, \quad \mathbf{y} = y_1 \chi_A + y_2 \chi_B,$$

where $B := \Omega \setminus A$. Put $a := \mu(A)$, $b := \mu(B)$. Since $\psi^{-1} = \gamma^{-1}$ is increasing, we have

$$\psi^{-1}(ax_1 + by_1) + \gamma^{-1}(ax_2 + by_2) > \psi^{-1}(x_1) + \gamma^{-1}(x_2).$$

Hence, as φ is decreasing,

$$\varphi(\psi^{-1}(ax_1 + by_1) + \gamma^{-1}(ax_2 + by_2)) < \varphi(\psi^{-1}(x_1) + \gamma^{-1}(x_2))$$

and, consequently,

$$\varphi(\psi^{-1}(ax_1 + by_1) + \gamma^{-1}(ax_2 + by_2)) < a\varphi(\psi^{-1}(x_1) + \gamma^{-1}(x_2)) + b\varphi(\psi^{-1}(y_1) + \gamma^{-1}(y_2)).$$

Since this inequality holds true for all $x_1, x_2, y_1, y_2 > 0$, replacing here x_1, x_2, y_1, y_2 , respectively by $\psi(x_1), \psi(x_2), \gamma(y_1), \gamma(y_2)$ and making again use of the decreasing monotonicity of φ , we obtain

$$\begin{aligned} \mathbb{P}_\psi(\mathbf{x}) + \mathbb{P}_\gamma(\mathbf{y}) &= \psi^{-1}(a\psi(x_1) + b\psi(x_2)) + \gamma^{-1}(a\gamma(y_1) + b\gamma(y_2)) \\ &\geq \varphi^{-1}(a\varphi(x_1 + y_1) + b\varphi(x_2 + y_2)) = \mathbb{P}_\varphi(\mathbf{x} + \mathbf{y}), \end{aligned}$$

which completes the proof. \square

4. The Pexider type generalization of the Minkowski inequality in case (II)

In this section we consider the case when the underlying measure space (Ω, Σ, μ) satisfies the condition (II) for every $A \in \Sigma$, we have $\mu(A) \leq 1$ or $\mu(A) = \infty$.

First we consider the case $\mu(\Omega) = 1$. We begin with

Theorem 4. Let (Ω, Σ, μ) be a measure space such that $\mu(\Omega) = 1$, $\Sigma = \{\emptyset, \Omega, A, \Omega \setminus A\}$ for some $A \subset \Omega$ and such that $0 < \mu(A) < 1$. Suppose that $\varphi, \psi, \gamma : (0, \infty) \rightarrow (0, \infty)$ are bijective functions.

(i) If φ is increasing, then inequality (P):

$$\mathbb{P}_\varphi(\mathbf{x} + \mathbf{y}) \leq \mathbb{P}_\psi(\mathbf{x}) + \mathbb{P}_\gamma(\mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in S_+(\Omega, \Sigma, \mu),$$

holds true if, and only if, the function $\Phi : (0, \infty)^2 \rightarrow (0, \infty)$ defined by

$$\Phi(s, t) := \varphi(\psi^{-1}(s) + \gamma^{-1}(t)), \quad s, t > 0, \quad (4.1)$$

is concave.

(ii) If φ is decreasing, then inequality (P) holds true if, and only if, the function Φ is $\mu(A)$ -convex, that iff

$$\Phi(as_1 + (1-a)s_2, at_1 + (1-a)t_2) \leq a\Phi(s_1, t_1) + (1-a)\Phi(s_2, t_2), \quad s_1, s_2, t_1, t_2 > 0,$$

where $a := \mu(A)$. If, moreover, ψ and γ are continuous, then Φ is convex.

Proof. (i) Put $a := \mu(A)$, $b := \mu(\Omega \setminus A) = 1 - a$. By Remark 1, the inequality (P) is equivalent to the inequality

$$\varphi^{-1}(a\varphi(x_1 + y_1) + b\varphi(x_2 + y_2)) \leq \psi^{-1}(a\psi(x_1) + b\psi(x_2)) + \gamma^{-1}(a\gamma(y_1) + b\gamma(y_2))$$

for all $x_1, y_1, x_2, y_2 > 0$. Replacing here x_i by $\psi^{-1}(x_i)$, y_i by $\gamma^{-1}(y_i)$ ($i = 1, 2$), by the increasing monotonicity of φ , we obtain the equivalent inequality

$$a\varphi(\psi^{-1}(x_1) + \gamma^{-1}(y_1)) + b\varphi(\psi^{-1}(x_2) + \gamma^{-1}(y_2)) \leq \varphi(\psi^{-1}(ax_1 + bx_2) + \gamma^{-1}(ay_1 + by_2))$$

for all $x_1, y_1, x_2, y_2 > 0$, which says that the function Φ is a -concave. By the Daróczy–Páles identity [10], the function Φ is Jensen concave. Since Φ is bounded from below, by Bernstein–Doetsch theorem (cf. [7]), it is continuous, and, consequently, Φ is concave.

(ii) We omit an analogous argument. \square

Remark 5. In the case (ii) when φ is decreasing, in view of Proposition 3, at least one of the functions ψ and γ must be increasing.

In a similar way one can prove

Theorem 5. Let $n \in \mathbb{N}$, $n \geq 2$ be fixed and (Ω, Σ, μ) be a measure space such that $\mu(\Omega) = 1$, $\Sigma = \{\emptyset, A_1, \dots, A_n\}$ for some pairwise disjoint sets $A_i \subset \Omega$, $i = 1, \dots, n$, of positive measures. Suppose that $\varphi, \psi_1, \dots, \psi_n : (0, \infty) \rightarrow (0, \infty)$ are bijective.

(i) If φ is increasing, then

$$\mathbb{P}_{\varphi}(\mathbf{x}_1 + \cdots + \mathbf{x}_n) \leq \mathbb{P}_{\psi_1}(\mathbf{x}_1) + \cdots + \mathbb{P}_{\psi_n}(\mathbf{x}_n), \quad \mathbf{x}_1, \dots, \mathbf{x}_n \in S_+(\Omega, \Sigma, \mu), \quad (4.2)$$

if, and only if, the function $\Phi : (0, \infty)^n \rightarrow (0, \infty)$ defined by

$$\Phi(t_1, \dots, t_n) := \varphi(\psi_1^{-1}(t_1) + \cdots + \psi_n^{-1}(t_n)), \quad t_1, \dots, t_n > 0, \quad (4.3)$$

is concave.

(ii) if φ is decreasing and ψ_1, \dots, ψ_n are continuous, then inequality (4.2) is, and only if, the function (4.3) is convex.

Lemma 5. Let $\varphi, \psi, \gamma : (0, \infty) \rightarrow (0, \infty)$ be bijective, twice differentiable, such that $\varphi' \psi' \gamma' \varphi'' \psi'' \gamma'' \neq 0$ in $(0, \infty)$.

(i) If $\varphi' > 0$ then Φ defined by (4.1) is concave if, and only if,

$$\frac{\psi''(s)}{\psi'(s)} \leq \frac{\varphi''(s+t)}{\varphi'(s+t)}, \quad \frac{\gamma''(t)}{\gamma'(t)} \leq \frac{\varphi''(s+t)}{\varphi'(s+t)}, \quad s, t > 0, \quad (4.4)$$

and

$$\frac{\varphi''(s+t)}{[\varphi'(s+t)]^3} \frac{\psi''(s)}{[\psi'(s)]^3} \frac{\gamma''(s)}{[\gamma'(s)]^3} \left(\frac{\varphi'(s+t)}{\varphi''(s+t)} - \frac{\psi'(s)}{\psi''(s)} - \frac{\gamma'(t)}{\gamma''(t)} \right) \geq 0, \quad s, t > 0. \quad (4.5)$$

(ii) If $\varphi' < 0$ then Φ defined by (4.1) is concave if, and only if, inequality (4.5) and the reversed inequalities (4.4) are fulfilled.

Proof. For arbitrarily fixed $s, t > 0$ and admissible $u, v \in \mathbb{R}$ define a function of a single variable

$$G(\tau) := \Phi(s + u\tau, t + v\tau) = \varphi(\psi^{-1}(s + u\tau) + \gamma^{-1}(t + v\tau)).$$

The function Φ is concave if, and only if, for all $s, t > 0$ and $u, v \in \mathbb{R}$ we have $G''(0) \leq 0$ (cf. [11, pp. 79–81]). Put $f := \varphi^{-1}$, $g := \psi^{-1}$, $h := \gamma^{-1}$. From the definition of G we have

$$f(G(\tau)) = g(s + u\tau) + h(t + v\tau)$$

for τ in a neighborhood of 0. Hence we obtain

$$f'(G(0))G'(0) = g'(s)u + h'(t)v$$

and

$$f''(G(0))[G'(0)]^2 + f'(G(0))G''(0) = g''(s)u^2 + h''(t)v^2.$$

Eliminating $G'(0)$ from these two equalities, we get

$$[f'(G(0))]^3 G''(0) = Au^2 - 2Buv + Cv^2, \quad (4.6)$$

where

$$A := g''(s)[f'(G(0))]^2 - f''(G(0))[g'(s)]^2,$$

$$B := f''(G(0))g'(s)h'(t),$$

$$C := h''(t)[f'(G(0))]^2 - f''(G(0))[h'(t)]^2.$$

If φ' is positive, then $f'(G(0)) > 0$. From (4.6) we conclude that $G''(0) \leq 0$ if

$$A \leq 0, \quad C \leq 0, \quad AC \geq B^2.$$

Note that

$$G(0) = \varphi(\psi^{-1}(s) + \gamma^{-1}(t)), \quad f' \circ \varphi = \frac{1}{\varphi'}, \quad f'' \circ \varphi = -\frac{\varphi''}{(\varphi')^3},$$

$$g' \circ \psi = \frac{1}{\psi'}, \quad g'' \circ \psi = -\frac{\psi''}{(\psi')^3}, \quad h' \circ \gamma = \frac{1}{\gamma'}, \quad h'' \circ \gamma = -\frac{\gamma''}{(\gamma')^3}.$$

It follows that, for all $s, t > 0$,

$$f'(G(0)) = f'(\varphi(\psi^{-1}(s) + \gamma^{-1}(t))) = \frac{1}{\varphi'(\psi^{-1}(s) + \gamma^{-1}(t))},$$

$$f''(G(0)) = f''(\varphi(\psi^{-1}(s) + \gamma^{-1}(t))) = -\frac{\varphi''(\psi^{-1}(s) + \gamma^{-1}(t))}{[\varphi'(\psi^{-1}(s) + \gamma^{-1}(t))]^3},$$

$$g'(s) = \frac{1}{\psi'(\psi^{-1}(s))}, \quad g''(s) = -\frac{\psi''(\psi^{-1}(s))}{[\psi'(\psi^{-1}(s))]^3},$$

$$h'(t) = \frac{1}{\gamma'(\gamma^{-1}(t))}, \quad h''(t) = -\frac{\gamma''(\gamma^{-1}(t))}{[\gamma'(\gamma^{-1}(t))]^3}.$$

Hence, by the definitions of A, B, C , and replacing, for the simplicity of notations, s by $\psi(s)$, t by $\gamma(t)$, we obtain

$$A \leq 0 \iff \frac{\varphi''(s+t)}{\varphi'(s+t)} \leq \frac{\psi''(s)}{\psi'(s)}, \quad C \leq 0 \iff \frac{\varphi''(s+t)}{\varphi'(s+t)} \leq \frac{\gamma''(t)}{\gamma'(t)},$$

and

$$AC \geq B^2 \iff \frac{\varphi''(s+t)}{[\varphi'(s+t)]^3} \frac{\psi''(s)}{[\psi'(s)]^3} \frac{\gamma''(t)}{[\gamma'(t)]^3} \left(\frac{\varphi'(s+t)}{\varphi''(s+t)} - \frac{\psi'(s)}{\psi''(s)} - \frac{\gamma'(t)}{\gamma''(t)} \right) \geq 0$$

for all $s, t > 0$, which completes the proof of the first part of the lemma.

Since the proof of the second part is analogous we omit it. \square

From this lemma we obtain the following

Proposition 5. Let $\varphi, \psi, \gamma : (0, \infty) \rightarrow (0, \infty)$ be bijective, twice differentiable with positive first and second derivatives in $(0, \infty)$. If

$$\frac{\psi'(s)}{\psi''(s)} + \frac{\gamma'(t)}{\gamma''(t)} \leq \frac{\varphi'(s+t)}{\varphi''(s+t)}, \quad s, t > 0, \quad (4.7)$$

then Φ defined by (4.1) is concave.

Proof. Since $\frac{\psi'}{\psi''}, \frac{\gamma'}{\gamma''}, \frac{\varphi'}{\varphi''}$ are positive, (4.7) implies reversed inequalities (4.4) and (4.5) in the above lemma. \square

Remark 6. It is easy to formulate and prove the counterpart of the above proposition ensuring the concavity (convexity) of the function Φ defined by (4.3).

Theorem 6. Let (Ω, Σ, μ) be a measure space such that $\mu(\Omega) \leq 1$, $\Sigma = \{\emptyset, \Omega, A, \Omega \setminus A\}$ for some $A \subset \Omega$ and such that $0 < \mu(A) < 1$. Suppose that $\varphi, \psi, \gamma : (0, \infty) \rightarrow (0, \infty)$ are increasing bijective functions.

If φ is increasing and the function $\Phi : (0, \infty)^2 \rightarrow (0, \infty)$ defined by

$$\Phi(s, t) := \varphi(\psi^{-1}(s) + \gamma^{-1}(t)), \quad s, t > 0,$$

is concave, then inequality (P) holds true.

Proof. The bijectivity and increasing monotonicity of the functions φ, ψ, γ imply that $\varphi(0+) = \psi(0+) = \gamma(0+) = 0$. Therefore we may assume that $\varphi(0) = \psi(0) = \gamma(0) = 0$. Then the function Φ_0

$$\Phi_0(s, t) := \varphi(\psi^{-1}(s) + \gamma^{-1}(t)), \quad s, t \geq 0$$

is well defined on \mathbb{R}_+^2 . Since Φ_0 is continuous and $\Phi := \Phi_0|_{(0, \infty)^2}$ is concave, it is obvious that Φ_0 is concave in \mathbb{R}_+^2 , and $\Phi_0(0, 0) = 0$.

Take arbitrary $\mathbf{x}, \mathbf{y} \in S_+(\Omega, \Sigma, \mu)$. Then there exist $x_1, x_2, y_1, y_2 > 0$ such that

$$\mathbf{x} = x_1 \chi_A + x_2 \chi_B, \quad \mathbf{y} = y_1 \chi_A + y_2 \chi_B,$$

where $B := \Omega \setminus A$. Putting $a := \mu(A)$, $b := \mu(B)$, by the concavity of Φ_0 we have

$$a\varphi(\psi^{-1}(u_1) + \gamma^{-1}(v_1)) + b\varphi(\psi^{-1}(u_2) + \gamma^{-1}(v_2)) + (1-a-b)\varphi(\psi^{-1}(0) + \gamma^{-1}(0)) \\ \leq \varphi(\psi^{-1}(au_1 + bu_2 + (1-a-b)0)) + \gamma^{-1}(av_1 + bv_2 + (1-a-b)0),$$

whence, for all $u_1, u_2, v_1, v_2 > 0$,

$$a\varphi(\psi^{-1}(u_1) + \gamma^{-1}(v_1)) + b\varphi(\psi^{-1}(u_2) + \gamma^{-1}(v_2)) \leq \varphi(\psi^{-1}(au_1 + bu_2) + \gamma^{-1}(av_1 + bv_2)).$$

Setting here $u_1 := \varphi(x_1)$, $u_2 := \varphi(x_2)$, $v_1 := \varphi(y_1)$, $v_2 := \varphi(y_2)$, and making use of the increasing monotonicity of φ , we obtain

$$\varphi^{-1}(a\varphi(x_1 + y_1) + b\varphi(x_2 + y_2)) \leq \psi^{-1}(a\psi(x_1) + b\psi(x_2)) + \gamma^{-1}(a\gamma(y_1) + b\gamma(y_2)),$$

that is

$$\mathbb{P}_\varphi(\mathbf{x} + \mathbf{y}) \leq \mathbb{P}_\varphi(\mathbf{x}) + \mathbb{P}_\varphi(\mathbf{y}),$$

which completes the proof. \square

Remark 7. Applying similar reasoning, one can easily get the suitable version of this result in the case when φ is decreasing and the counterparts of both results in the case when Σ is a finite family of sets. We omit their easy formulations.

Remark 8. The obtained generalizations of the Minkowski inequality allow to introduce some new paranormed function spaces (cf. [9]).

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